

GÖDEL'S PROOF: A REVISIONIST VIEW

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ABSTRACT. This note presents a revised assessment of Gödel's proof. I show that the proof can be modified to establish that there exists a system P' such that: either P' is inconsistent (and P is also inconsistent) or P' is consistent and yet has no model. To define P' I firstly present a semantics for Gödel's P , and then define a theory P_0 which is syntactically identical to P however the meaning of the type one variables differs (assuming P is consistent) in that for any interpretation of P_0 these variables range over all and only the individuals assigned to the P_0 numerals. P' is the theory obtained by adding the negation of a Gödel sentence for P_0 to the proper axioms of P_0 .

1. INTRODUCTION

This note presents a revised assessment of Gödel's proof. I show that the proof can be modified to establish that there exists a system P' such that: either P' is inconsistent (and P is also inconsistent) or P' is consistent and yet has no model. To define P' I firstly present a semantics for Gödel's P , and then define a theory P_0 which is syntactically identical to P however the meaning of the type one variables differs (assuming P is consistent) in that for any interpretation of P_0 these variables range over all and only the individuals assigned to the P_0 numerals. P' is the theory obtained by adding the negation of a Gödel sentence for P_0 to the proper axioms of P_0 . To simplify the notation I generally avoid the use of quotation or quasi-quotation symbols even when precision would be improved by their use, since this is unlikely to give rise to confusion in the following. I generally follow the symbolism of [2] though I adapt the symbolism of [3] in defining semantic notions. The following section describes the system of interest P' .

2. THE SYSTEM P'

For brevity I assume familiarity with Gödel's system P [2]. To describe the system of interest P' I firstly present a semantics for P - that is, a description of how to informally define the notions of an interpretation of P , the 'truth' of a P formula under an interpretation and so on. The account is essentially a generalisation of the first-order definitions presented in [3] (with Definitions 3-4 involving paraphrase). I avoid however any suggestion that the metatheoretical notions can be formalised in an orthodox set theory or extension of P (c.f. [4]). To present this account some definitions are used. Firstly the following notion of an interpretation M of the language of P is used:

Definition 1. *An interpretation M of P shall consist of the following:*

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- (1) The domain D of the interpretation, being a set of objects a, b, \dots (informally, the individuals over which the type one P variables ' x_1 ', ' y_1 ', \dots range under the interpretation M).
- (2) An assignment of some fixed individual $(0)^M$ in D to the P constant ' 0 ' (the numeral zero).
- (3) An assignment of an operation $(f)^M$ defined on D to the P constant ' f ' (the symbol for the successor function).

Secondly, the definition of the following classes and sequences of objects / classes are used:

Definition 2. For M an interpretation of P , the classes $\sigma_1, \dots, \sigma_n, \dots, \sigma, \Sigma$ and denumerable sequences s^1, \dots, s^m, \dots are defined as follows:

- (1) σ_1 shall be D the domain of M and for a natural number $n > 1$, σ_n shall be the power set of σ_{n-1} :

$$(2.1) \quad \sigma_n = \begin{cases} D & \text{if } n = 1 \\ \wp(\sigma_{n-1}) & \text{if } n > 1 \end{cases}$$

- (2) For a natural number $m > 1$, s^m shall be a denumerable sequence of elements in σ_m : $s_1^m, \dots, s_n^m, \dots$;
- (3) Σ shall be the class of all σ such that: σ includes one, and only one, sequence s^j for each natural number $j > 0$ and nothing else.

Thirdly, the notion of a P formula c being 'satisfied' at a certain set σ , as defined above, is used (c.f. [3]):

Definition 3. For M an interpretation of P , the P formula c shall be satisfied at σ in Σ if and only if:

- (1) c is an elementary formula $a_n(b_{n-1})$ and (with a_n the i th type n variable):
 - (a) For $n > 2$, with b_{n-1} the j th type $n-1$ variable, we have (for s^n, s^{n-1} in σ): s_j^{n-1} is in s_i^n ;
 - (b) For b_{n-1} the numeral \overline{m} we have (for s^2, s^1 in σ): the object assigned to the numeral \overline{m} under M is in s_i^2 ;
 - (c) Otherwise, where b_{n-1} consists of a sequence of t ' f 's followed by the j th type 1 variable v we have (for s^2, s^1 in σ): the object $(f)^M[\dots[s_j^1]]$ - where t applications of the operation $(f)^M$ are involved - is in s_i^2 ;
- (2) c is a formula $\sim (a)$ and a is not satisfied at σ ;
- (3) c is a formula $(a) \vee (b)$ and σ satisfies a or σ satisfies b ;
- (4) c is a formula $a_n \Pi(b)$ (with a_n the i th type n variable): for every σ' in Σ which differs from σ in at most the i th element of the sequence s^n of elements from σ_n , b is satisfied at σ' .

Finally, the required notions of truth, and falsity of a P formula under an interpretation, and of an interpretation being a model of P are defined as follows (c.f. [3]):

Definition 4. For M an interpretation of P , and b a P formula we have::

- (1) b is true under M iff b is satisfied at every σ in Σ ;
- (2) b is false under M iff b is not satisfied at any σ in Σ ;
- (3) M is a model of P iff every P theorem is true under M .

To simplify the presentation I assume where convenient that the above notions provide an adequate account of the notions of truth and falsity for the language of P (if any such account is possible) and proceed directly to a description of the system P_0 . The syntax of P_0 is identical to that of P and the semantics differ formally with respect to just one point, namely Definition 2 Part 1. (Strictly speaking, the statement in parenthesis at Definition 1 Part 1 should also be replaced with the following: 'informally, the individuals assigned to the P numerals '0', 'f0', ... under the interpretation M '.) The replacement clause of Definition 2 Part 1 for P_0 is as follows (the equation is omitted for brevity): ' σ_1 shall be the smallest subset of D the domain of M that includes $(0)^M$ and is closed with respect to the operation $(f)^M$ - hence σ_1 includes only the individuals assigned to the numerals under M $((0)^M, (f)^M[(0)^M], \dots)$ - and for natural number $n > 1$, σ_n shall be the power set of σ_{n-1} '.

We now come to the system of interest P' . Since the syntax of P_0 and P is identical, the Gödel sentence of both systems is the same - for brevity $x_1\Pi(r)$. P' is obtained from P_0 by adding the negation of this formula, $\sim(x_1\Pi(r))$, as a proper axiom. For P' thus defined, we therefore have the following result:

Proposition 1. *P' is either inconsistent (and P is also inconsistent) or P' is consistent and yet has no model.*

Proof. The proof of Proposition 1 is by cases as follows.

P' is consistent: By [2] Theorem VI, the hypothesis that P_0 is consistent implies (using $\not\vdash_{P_0} / \vdash_{P_0}$ for it is / not provable in P_0 etc that):

- (1) $\not\vdash_{P_0} x_1\Pi(r)$ and
- (2) For any P' numeral \bar{n} : $\vdash_{P_0} \text{Subst} \left(r \frac{x_1}{\bar{n}} \right)$.

Since P' is an extension of P_0 however 2 implies, given the semantics for P' , that $x_1\Pi(r)$ is true in every model of P' . Yet since $\sim(x_1\Pi(r))$ is a proper axiom of P' , $\sim(x_1\Pi(r))$ is also true in every model of P' . Thus by the law of contradiction if P' is consistent it has no models.

P' is inconsistent: If P' is inconsistent then, by the law of contradiction, it has no models. But if P' is inconsistent then $x_1\Pi(r)$ must be provable in P_0 which implies that it is also provable in P and hence that P is also inconsistent.

□

I turn now to a brief discussion of the significance of these results.

3. CONCLUSIONS

A discussion that relates the above result to the large literature on the Gödel phenomenon is beyond the aims of this note. The more limited aim of the discussion is rather to sketch how the the main arguments of [2] might be recast to assimilate the above results to the framework that Gödel employs. Gödel's [2] framework may be roughly characterised as the metamathematical investigation of the very broad class of systems of interest, conducted in the first instance through a study of the system P . Viewed from this perspective, the main conclusion established above does not strictly speaking constitute a paradox: from a purely metamathematical perspective, neither the inconsistency of P nor the existence of a consistent

theory that lacks a model constitute a contradiction in themselves. Viewed in the broader context however the analysis provides strong evidence that the formalist metatheoretic reasoning about such systems is subject to paradox, so that the formalist approach fails to provide an adequate account of classical mathematics. In relation to first-order Peano arithmetic, for example, the demonstration yields the absurd conclusion that the theory is inconsistent [1]. This evidence from the case of first-order arithmetic is also important because the semantic metatheory used in that case involves only a minor revision of the orthodox semantics. While the theory of truth for P presented above involves some novelties, the comparison with the first-order case shows that the analysis does is not based solely or primarily on innovations in the account of truth used.

REFERENCES

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